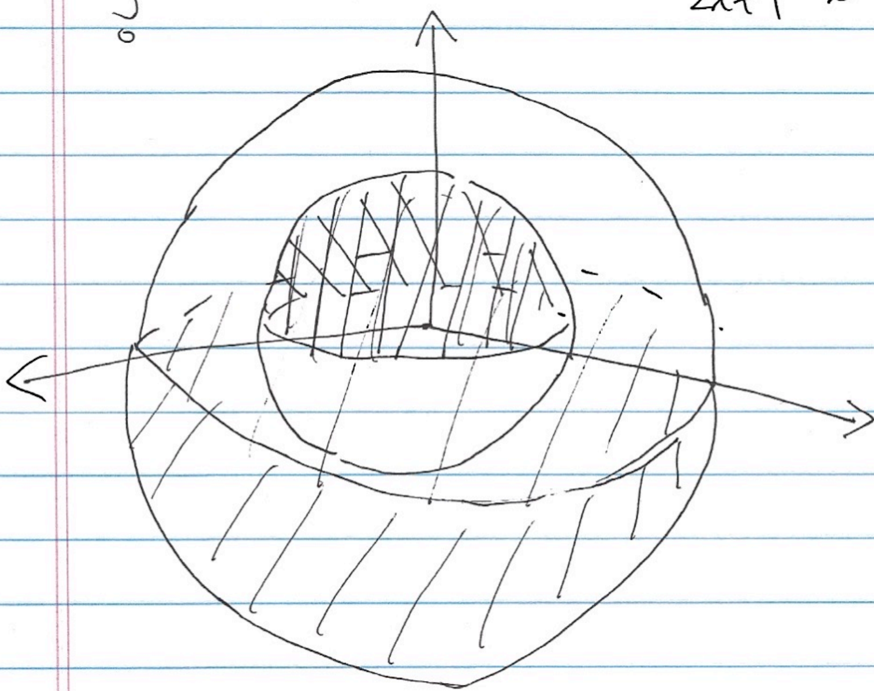
Jackson

3.1 $\langle P_l | P_{l'} \rangle = \int_{-1}^{1} P_l[x] P_{l'}^*[x]\, dx = \frac{2}{2l+1}$.

$$\Phi = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l[\cos\theta].$$

$$\int_0^{\pi} \Phi(\theta, r) P_l^*[\cos\theta] \sin\theta\, d\theta = \frac{2}{2l+1}[A_l r^l + B_l r^{-(l+1)}].$$



Boundary condition given by $V(r,\theta) = \begin{cases} V & r=a,\ 0 \le \theta \le \frac{\pi}{2} \\ & r=b,\ \frac{\pi}{2} < \theta \le \pi \\ 0 & \text{otherwise.} \end{cases}$

$$\Rightarrow \int_0^{\pi} \Phi(r,\theta) P_l^*[\cos\theta] \sin\theta\, d\theta$$

$$\Rightarrow \quad V\int_0^{\pi/2} P_l[\cos\theta] \sin\theta\, d\theta = \frac{2}{2l+1}[A_l a^l + B_l a^{-(l+1)}]$$

$$V\int_{\pi/2}^{\pi} P_l[\cos\theta] \sin\theta\, d\theta = \frac{2}{2l+1}[A_l b^l + B_l b^{-(l+1)}]$$

It's now more convenient to let $x = \cos\theta$

$$\Rightarrow \quad V \int_0^1 P_l[x]\, dx = \frac{2}{2l+1} [A_l a^l + B_l a^{-(l+1)}]$$

$$V \int_{-1}^{0} P_l[x]\, dx = \frac{2}{2l+1} [A_l b^l + B_l b^{-(l+1)}]$$

Use $\quad P_l = \frac{1}{2l+1} \left[\frac{d}{dx} P_{l+1} - \frac{d}{dx} P_{l-1} \right]$

$$V [P_{l+1} - P_{l-1}] \Big|_0^1 = 2 [A_l a^l + B_l a^{-(l+1)}],$$

$$V [P_{l+1} - P_{l-1}] \Big|_{-1}^{0} = 2 [A_l b^l + B_l b^{-(l+1)}].$$

$$V [P_{l+1} - P_{l-1}] \Big|_0^1 = V [P_{l+1}[1] - P_{l-1}[1] - P_{l+1}[0] + P_{l-1}[0]]$$

(Use $P_l[1] = 1$) $\quad = V [P_{l-1}[0] - P_{l+1}[0]].$

$$V [P_{l+1} - P_{l-1}] \Big|_{-1}^{0} = V [P_{l+1}[0] - P_{l-1}[0] - P_{l+1}[-1] + P_{l-1}[-1]]$$

(Use $P_l[-1] = (-1)^l$) $\quad = V [P_{l+1}[0] - P_{l-1}[0]]$

Introducing $\xi_l \equiv P_{l-1}[0] - P_{l+1}[0]$, we have

$$V \xi_l = 2 [A_l a^l + B_l a^{-(l+1)}]$$

$$-V \xi_l = 2 [A_l b^l + B_l b^{-(l+1)}]$$

$$B_\ell \left\{ [a^{-(\ell+1)} - b^{-(\ell+1)}] - [a^{-(\ell+1)} + b^{-(\ell+1)}] \frac{[a^\ell - b^\ell]}{[a^\ell + b^\ell]} \right\} = V \xi_\ell$$

$$\boxed{A_\ell = -B_\ell \frac{[a^{-(\ell+1)} + b^{-(\ell+1)}]}{[a^\ell + b^\ell]}}$$

$$\boxed{B_\ell = V \xi_\ell \left\{ [a^{-(\ell+1)} - b^{-(\ell+1)}] - [a^{-(\ell+1)} + b^{-(\ell+1)}] \frac{[a^\ell - b^\ell]}{[a^\ell + b^\ell]} \right\}^{-1}.}$$

$\Uparrow$

General formula for A_ℓ, B_ℓ, for $\ell > 0$.

For $\ell = 0$,
$$V = 2[A_0 + B_0 a^{-1}]$$
$$= 2[A_0 + B_0 b^{-1}]$$

$$\Rightarrow B_0 = 0, \quad \cancel{A_\ell =} \; A_0 = V/2.$$

Since $\xi_{2n} = 0$ for all $n \geq 1$,

it's clear that $A_2 = B_2 = 0$,
$A_4 = B_4 = 0$,

that is, only odd terms of A, B show up,
except for $A_0 = V/2$.

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